Applications of Gröbner Fans

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Gröbner basis computation: the Gröbner Walk

Cones, Fans and Ideals

Idea: Incremental approach to Gröbner basis computation, based on polyhedral geometry.

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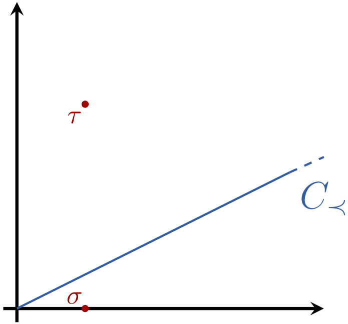
Let I be a fixed ideal in $\mathbb{Q}[x_1, \dots x_n]$. Each Gröbner basis $G_\prec = \{g_1, \dots g_s\}$ of I is associated to a polyhedral cone $C_\prec \subset \mathbb{R}^n$.

$$\left\{ \begin{array}{c} \text{Initial ideals} \\ in_\prec(I) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{marked Gröbner bases} \\ G_\prec \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Gröbner cones} \\ C_\prec \end{array} \right\}.$$

The cones form a **polyhedral fan**, called the *Gröbner Fan* of I.

The standard Gröbner walk

Task: Given a starting Gröbner basis $G_{\prec}$ and a target ordering $\prec'$, compute $G_{\prec'}$



Strategy: Starting from $C_{\prec}$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

Lemma 0 (Monomial order matrices)

Lemma

Let $\prec$ *be a monomial order on* $\mathbb{Q}[x_1, ..., x_n]$ *and* $A \in \mathbb{Q}^{k,n}$ *be a monomial order matrix of* $\prec$. *Let* $a_1 \in \mathbb{Q}^n_{\geq 0}$ *denote the first row of* A. *Then* $\prec$ *refines* a_1, *i.e. :*

$$\langle a_1, \beta \rangle < \langle a_1, \alpha \rangle \quad \Longrightarrow x^\beta \prec x^\alpha \;\; \textit{for all } \alpha, \beta \in \mathbb{N}^n.$$

In particular: $a_1 \in C_\prec$ *holds for any ideal* $I \lhd \mathbb{Q}[x_1, ..., x_n]$.

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This means that σ and τ are already known a priori!
We use this informatio + $C_\prec$ to compute a boundary point of $C_\prec$

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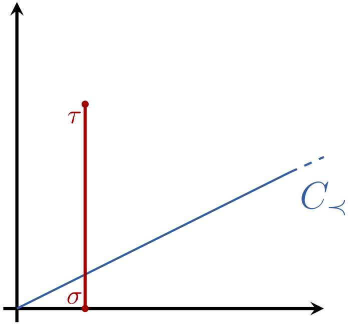

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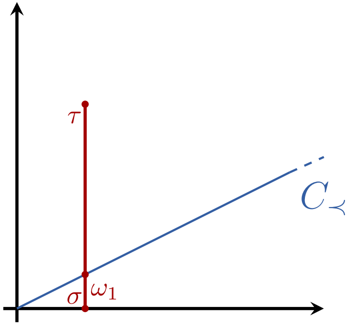

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Lemma 1

A **Marked** Gröbner basis is a reduced Gröbner basis with the leading terms identified.

Lemma

Let I be an ideal and $\prec$ a monomial order. For $G_\prec = \{g_1, ..., g_r\}$ and $\omega \in C_\prec \cap \mathbb{Q}^n_{\geq 0}$, the set

$$in_\omega(G_\prec) = \{in_\omega(g_1), ..., in_\omega(g_r)\}$$

is the marked Gröbner basis of $in_\omega(I)$ w.r.t $\prec$.

This holds in particular when $\omega \in \partial C_\prec$. So given ω and $G_\prec$, we obtain a basis of $in_\omega(I)$ "for free".

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This holds in particular when $\omega \in \partial C_\prec$. So given ω and $G_\prec$, we obtain a basis of $in_\omega(I)$ "for free". The basis of $in_\omega(I)$ corresponds to the lower-dimensional cone in the Gröbner fan!

Lemma 2 (Lifting)

Lemma

Let H be the Gröbner basis of I with respect to a term ordering $<$ and let $\omega \in \mathbb{Q}^n$ lie on the boundary of the cone in the Gröbner fan $\mathbb{G}(I)$ corresponding to $<$.
If $M = \{m_1, ..., m_r\}$ is the marked Gröbner basis of $in_\omega(I)$ with respect to the refinement ordering $\prec'_\omega$, then

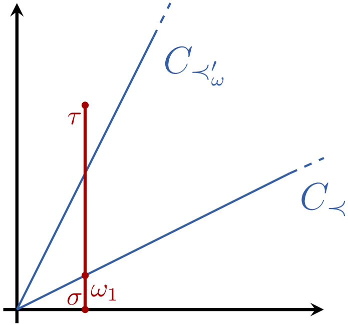
$$G := \{m_1 - \overline{m_1}^H, ..., m_r - \overline{m_r}^H\}$$

is a Gröbner basis of I with respect to $\prec'_\omega$.

This process is known as “lifting”

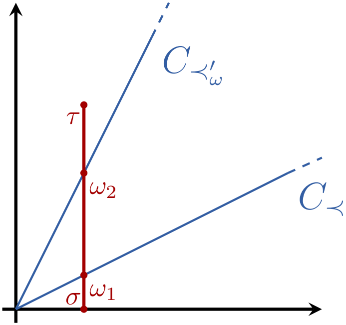
The standard Gröbner walk

If τ lies in our new cone, we're done!



If not, update the starting cone/basis and go again!

The standard Gröbner walk



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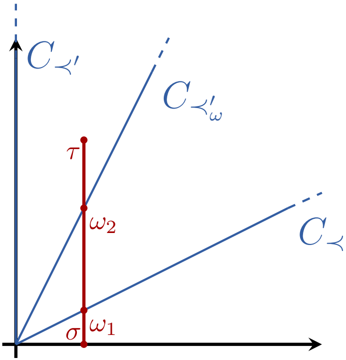


My Contribution

From this...

Algorithm 1 $\text{StandardGroebnerWalk}(G_{\prec}, A_{\prec}, A_{\prec'})$

Input: $G_{\prec}$, $A_{\prec}$ and $A_{\prec'}$

Output: $G_{\prec'}$

$\sigma \leftarrow (A_{\prec})_{1,\cdot}$
$\tau \leftarrow (A_{\prec'})_{1,\cdot}$
done $\leftarrow$ "False"
while done = "False" **do**
  $\omega \leftarrow \text{GetNextW}(G_{\prec}, \sigma, \tau)$
  $G' \leftarrow \text{Lift}(G_{\prec}, \omega, \tau)$
  $G' \leftarrow \text{Reduce}(G')$
  if $\omega = \tau$ **then**
    done $\leftarrow$ "True"
  else
    $\sigma \leftarrow \omega$
    $G_{\prec} \leftarrow G'$
    $A_{\prec} \leftarrow A_{\prec'_{\omega}}$
  end if
end while
return G'

To this!

Oscar.jl / experimental / GroebnerWalk / src / **common.jl**

Code Blame 138 lines (110 loc) · 4.49 KB

```julia
    lex([x, y])

```
"""
function groebner_walk(
  I::MPolyIdeal,
  target::MonomialOrdering = lex(base_ring(I)),
  start::MonomialOrdering = default_ordering(base_ring(I));
  algorithm::Symbol = :standard
)
  if algorithm == :standard
    walk = (x) -> standard_walk(x, target)
  elseif algorithm == :generic
    walk = (x) -> generic_walk(x, start, target)
  else
    throw(NotImplementedError(:groebner_walk, algorithm))
  end

  Gb = groebner_basis(I; ordering=start, complete_reduction=true)
  Gb = walk(Gb)

  return Oscar.IdealGens(Gb, target; isGB=true)
end

```

https://github.com/oscar-system/Oscar.jl/
```

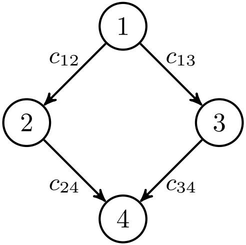
OSCAR demonstration

fpnowell.github.io

A Gröbner fan in the wild

Combinatorics of shortest paths

Let $\mathcal{G} = (V, E)$ be a directed acyclic graph (DAG) with non-negative edge weights c_{ij} for $i \to j \in E$.



$$C = \begin{pmatrix} -\infty & c_{12} & c_{13} & -\infty \\ -\infty & -\infty & c_{24} & -\infty \\ -\infty & -\infty & -\infty & c_{34} \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

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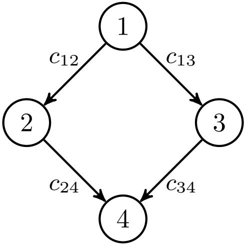

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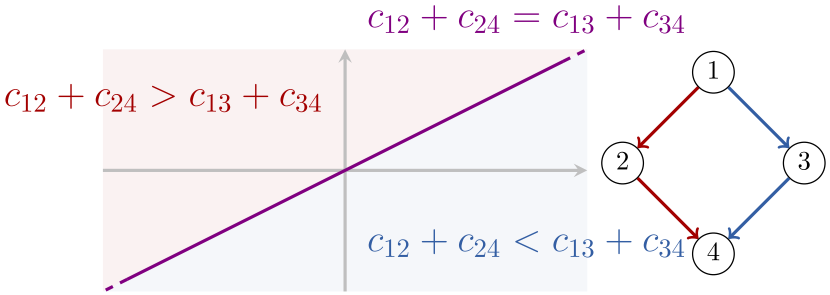
TASK: Classify the sets of edge weights which give rise to the same max-weighted paths.
(The *weight* of a directed path from i to j is the *sum* of the weights of its edges.)

It's a fan!



In the space of all matrices $\mathbb{R}[E]$ supported on $\mathcal{G}$ (this is $\mathbb{R}^4_{\geq 0}$), the matrices giving rise to the same max-weighted paths lie in a *cone*!

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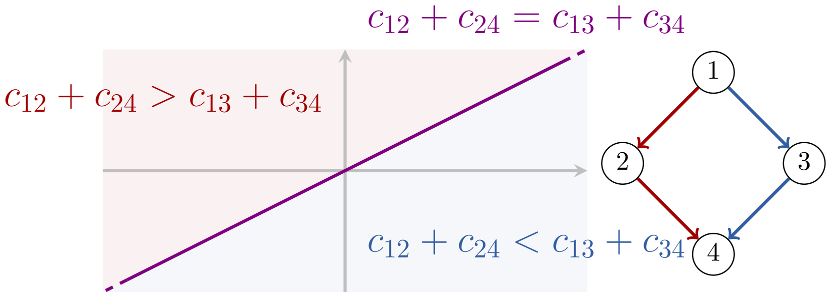

In the space of all matrices $\mathbb{R}[E]$ supported on $\mathcal{G}$ (this is $\mathbb{R}^4_{\geq 0}$), the matrices giving rise to the same max-weighted paths lie in a *cone*! The cones form a *polyhedral fan*!!

It's a Gröbner fan!

Let $P(i,j)$ denote the set of all directed paths from i to j The Gröbner fan of $G = (V, E)$ is

$$I_{\mathcal{G}} = \langle \sum_{\pi \in P(i,j)} \prod_{k_1 \to k_2 \in \pi} c_{k_1,k_2} : i,j \in V, |P(i,j)| > 1 \rangle \triangleleft \mathbb{R}[E].$$

In the example: $I_{\mathcal{G}} = \langle c_{12}c_{24} + c_{13}c_{34} \rangle$.

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Bottom line: All of the different possible structures of max-weighted paths can be computed with a Gröbner fan computation!