The GroebnerWalk package

Francesco Nowell

TU Berlin

29.07.2024

What is the Gröbner walk?

Motivating example

Given the ideal in $\mathbb{Q}[x, y, z]$

$$I = \langle\, 6+3x^3+16x^2z+14x^2y^3,\;\; 6+y^3z+17x^2z^2+7xy^2z^2+13x^3z^2 \,\rangle$$

compute the reduced Gröbner basis of I *w.r.t*
the lexicographic ordering with $x \succ y \succ z$

Motivating example

Given the ideal in $\mathbb{Q}[x, y, z]$

$$I = \langle\, 6+3x^3+16x^2z+14x^2y^3,\;\; 6+y^3z+17x^2z^2+7xy^2z^2+13x^3z^2 \,\rangle$$

compute the reduced Gröbner basis of I *w.r.t*
the lexicographic ordering with $x \succ y \succ z$

Problem: Calling Buchberger's algorithm runs into S-pairs of degree up to 46, with up to 466 terms and coefficients of order up to 10^{590}. This is slow!

Motivating example

Given the ideal in $\mathbb{Q}[x, y, z]$

$$I = \langle\, 6 + 3x^3 + 16x^2z + 14x^2y^3,\ \ 6 + y^3z + 17x^2z^2 + 7xy^2z^2 + 13x^3z^2 \,\rangle$$

compute the reduced Gröbner basis of I *w.r.t*
the lexicographic ordering with $x \succ y \succ z$

Problem: Calling Buchberger's algorithm runs into S-pairs of degree up to 46, with up to 466 terms and coefficients of order up to 10^{590}. This is slow!

Idea: Replace this one 'heavy' computation with many 'light' ones.

Gröbner cones

Let $I \lhd \mathbb{Q}[x_1, \ldots, x_n]$ be a fixed ideal and $\prec$ be a monomial ordering. We denote the marked Gröbner basis of I by

$$G_{\prec} = \{g_1, g_2, \ldots, g_s\} .$$

Gröbner cones

Let $I \triangleleft \mathbb{Q}[x_1, \ldots, x_n]$ be a fixed ideal and $\prec$ be a monomial ordering. We denote the marked Gröbner basis of I by

$$G_{\prec} = \{g_1, g_2, \ldots, g_s\}.$$

The marked Gröbner bases of I are in 1-1 correspondence with full-dimensional cones in $\mathbb{R}^n$. We call these **Gröbner cones** and refer to them by $C_{\prec}$.

Gröbner cones

Let $I \lhd \mathbb{Q}[x_1, \ldots, x_n]$ be a fixed ideal and $\prec$ be a monomial ordering. We denote the marked Gröbner basis of I by

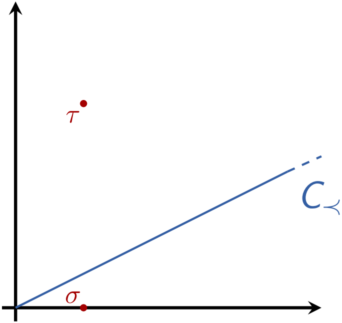
$$G_{\prec} = \{g_1, g_2, \ldots, g_s\}.$$

The marked Gröbner bases of I are in 1-1 correspondence with full-dimensional cones in $\mathbb{R}^n$. We call these **Gröbner cones** and refer to them by $C_{\prec}$.

The Gröbner cones of I form a polyhedral fan, called the **Gröbner fan**.

The standard Gröbner walk ([CKM97])

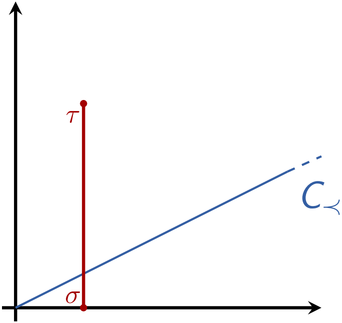
Task: Given monomial orders $\prec'$ and $\prec$ and $G_{\prec}$, compute $G_{\prec'}$.



Strategy: Starting from $C_{\prec}$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

The standard Gröbner walk ([CKM97])

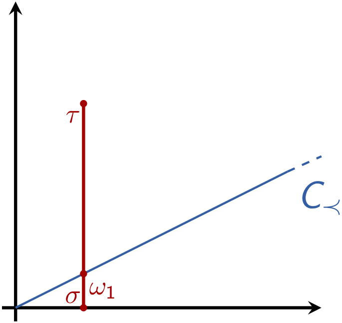
Task: Given monomial orders $\prec'$ and $\prec$ and $G_{\prec}$, compute $G_{\prec'}$.



Strategy: Starting from $C_{\prec}$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

The standard Gröbner walk ([CKM97])

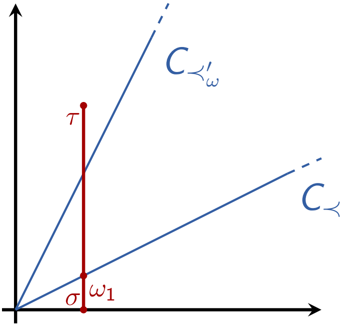
Task: Given monomial orders $\prec'$ and $\prec$ and $G_{\prec}$, compute $G_{\prec'}$.



Strategy: Starting from $C_{\prec}$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

The standard Gröbner walk ([CKM97])

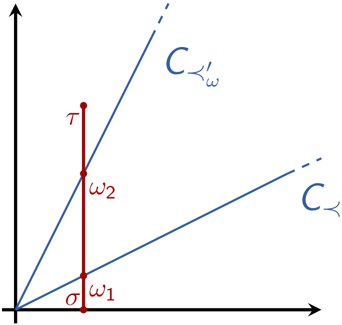
Task: Given monomial orders $\prec'$ and $\prec$ and $G_{\prec}$, compute $G_{\prec'}$.



Strategy: Starting from $C_{\prec}$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

The standard Gröbner walk ([CKM97])

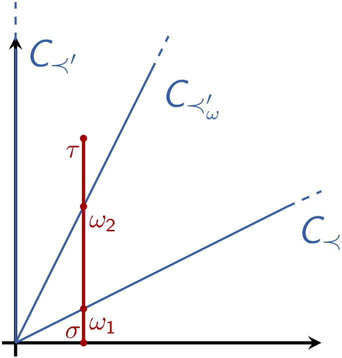
Task: Given monomial orders $\prec'$ and $\prec$ and $G_{\prec}$, compute $G_{\prec'}$.



Strategy: Starting from $C_{\prec}$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

The standard Gröbner walk ([CKM97])

Task: Given monomial orders $\prec'$ and $\prec$ and $G_\prec$, compute $G_{\prec'}$.



Strategy: Starting from $C_\prec$, 'walk' to $C_{\prec'}$, computing every intermediate Gröbner basis along the way.

Modifications to the Gröbner walk

The 'Zig-Zag' walk([AGK97]): scale the weight vectors at every step so they have integer entries.

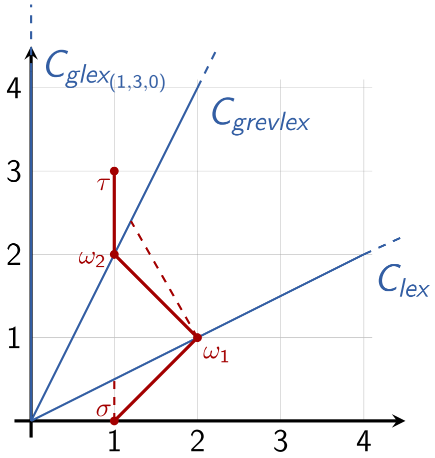
The perturbed walk([Tra00]): Modify $\sigma \in C_{\prec}$ and $\tau \in C_{\prec'}$ such as to ensure intersections on *facets* of cones.

The generic walk([Fuk+07]): Replace $\overline{\sigma\tau}$ with a symbolic line segment.

The GroebnerWalk package

OSCAR demonstration

https://zenodo.org/records/11451243 ([FN24])
https://github.com/ooinaruhugh/GroebnerWalk.jl



Comparison with groebner_basis

Ideal	Runtime in OSCAR (s., avg.)					
	groebner_basis		Standard walk		Generic walk	
	$\mathbb{Q}$	$\mathbb{F}_p$	$\mathbb{Q}$	$\mathbb{F}_p$	$\mathbb{Q}$	$\mathbb{F}_p$
cyclic5	0.07	0.05	0.08	0.07	0.91	0.879
cyclic6	5928.51	0.24	0.93	0.61	17.85	28.41
agk4	1407.76	$4 * 10^{-5}$	23.79	5.33	214.75	201.69
newellp1	2081.76	0.40	25.51	17.33	6042.97	4597.38
randomknap4	0.16	0.14	97.30	77.38	19.98	49.81

References I

[AGK97] Beatrice Amrhein, Oliver Gloor, and Wolfgang Küchlin. "On the walk". en. In: *Theoretical Computer Science* 187.1-2 (Nov. 1997), pp. 179–202. ISSN: 03043975. DOI: 10.1016/S0304-3975(97)00064-9. URL: https://linkinghub.elsevier.com/retrieve/pii/S0304397597000649 (visited on 09/01/2023).

[CKM97] S. Collart, M. Kalkbrener, and D. Mall. "Converting Bases with the Gröbner Walk". en. In: *Journal of Symbolic Computation* 24.3-4 (Sept. 1997), pp. 465–469. ISSN: 07477171. DOI: 10.1006/jsco.1996.0145. URL: https://linkinghub.elsevier.com/retrieve/pii/S0747717196901455 (visited on 09/01/2023).

[FN24] Kamillo Ferry and Francesco Nowell. *GroebnerWalk.jl: v1.0.1*. June 2024. DOI: 10.5281/ZENODO.11451243. URL: https://zenodo.org/doi/10.5281/zenodo.11451243 (visited on 07/06/2024).

References II

[Fuk+07] K. Fukuda et al. "The generic Gröbner walk". In: *Journal of Symbolic Computation* 42.3 (Mar. 2007), pp. 298–312. ISSN: 0747-7171. DOI: 10.1016/j.jsc.2006.09.004. URL: https://www.sciencedirect.com/science/article/pii/S0747717106001003 (visited on 09/01/2023).

[Tra00] Quoc-Nam Tran. "A Fast Algorithm for Gröbner Basis Conversion and its Applications". In: *Journal of Symbolic Computation* 30.4 (Oct. 2000), pp. 451–467. ISSN: 0747-7171. DOI: 10.1006/jsco.1999.0416. URL: https://www.sciencedirect.com/science/article/pii/S0747717199904169 (visited on 09/26/2023).