Polyhedral Geometry of Max-Linear Conditional Independence

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The work

Polyhedral Aspects of Maxoids (FN+25)

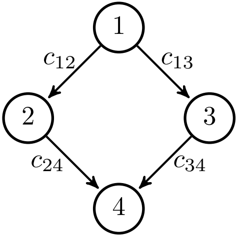
https://arxiv.org/abs/2504.21068

joint work with Tobias Boege, Kamillo Ferry, and Ben Hollering

Critical Paths and Polyhedral Geometry

Weighted DAGs

Let $\mathcal{G} = (V, E)$ be a Directed acyclic graph (DAG), equipped with edge weights $c_{ij} \in \mathbb{R}$ for $i \to j \in E$.



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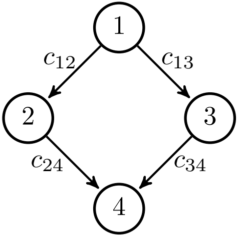

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$$C = \begin{pmatrix} -\infty & c_{12} & c_{13} & -\infty \\ -\infty & -\infty & -\infty & c_{24} \\ -\infty & -\infty & -\infty & c_{34} \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} \in \mathbb{R}^E \cong \mathbb{R}^4$$

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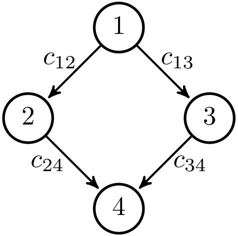

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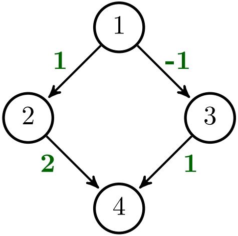
The *weight* of a directed path from i to j in $(\mathcal{G}, C)$ is

$$\omega_C(\pi) = \sum_{u \to v \in \pi} c_{uv}$$

Weighted DAGs: example



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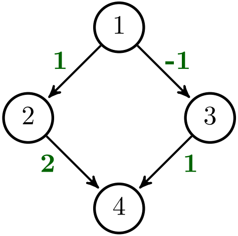

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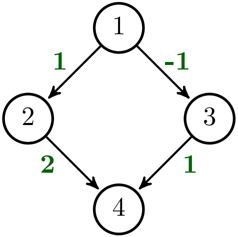

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$$\pi^{14}_{\text{crit}} = 1 \to 2 \to 4.$$

GOAL: For fixed C, characterize the set of all $C' \in \mathbb{R}^E$ which give rise to the same pairwise critical paths.

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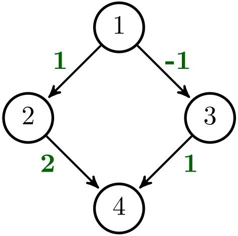

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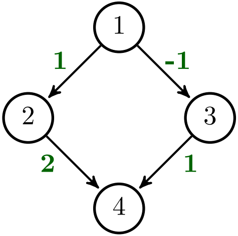

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Weighted DAGs:part II

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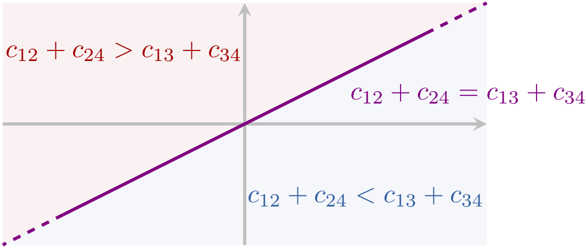

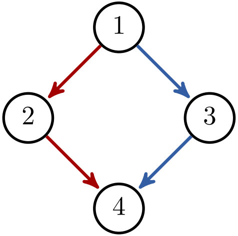

The critical path fan of $\mathcal{G}$

We call $C \in \mathbb{R}^E$ *generic* if no two paths $i \rightsquigarrow j$ have the same weight.

Theorem (FN+25, Theorem 2.9)

Let $(\mathcal{G}, C)$ be a weighted DAG with generic $C \in \mathbb{R}^E$. The set

$$\mathrm{cone}_{\mathcal{G}}(C) := \left\{ C' \in \mathbb{R}^E : \pi^{ij}_{crit}(\mathcal{G}, C) = \pi^{ij}_{crit}(\mathcal{G}, C') \ \forall i, j \in V \right\} \tag{1}$$

is a full-dimensional open polyhedral cone defined by linear inequalities of the form

$$\omega_{C'}(\pi^{ij}_{crit}(\mathcal{G}, C)) > \omega_{C'}(\pi), \quad \textit{for each } \pi : i \rightsquigarrow j,\ \pi \neq \pi^{ij}_{crit}(\mathcal{G}, C)). \tag{2}$$

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The cones form a complete polyhedral fan in $\mathbb{R}^E$, which we denote by $\mathcal{F}_{\mathcal{G}}$.

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Let $\mathcal{G} = (V, E)$ be a fixed DAG. For $i, j \in V$ consider the *path polynomial*

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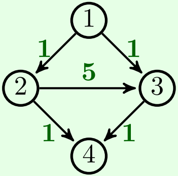
Example



$$f_{24} = x_{24} + x_{23}x_{34}$$
$$F_{24} = \max\{x_{24}, x_{23} + x_{34}\}$$
$$F_{24}(C) = \max\{c_{24}, c_{23} + c_{34}\} = \max\{\mathbf{1}, \mathbf{5+1}\}$$

Tropical hypersurfaces

Every tropical polynomial $F \in \mathbb{R}[x_1, \dots x_n]$ defines a *tropical hypersurface* $\mathcal{T}(F)$, which in turn induces a *polyhedral subdivision* of $\mathbb{R}^n$.

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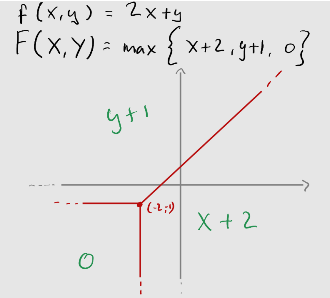
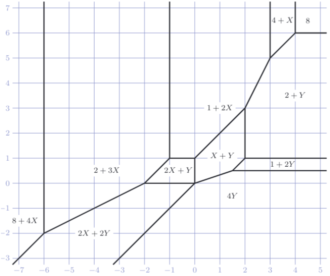


Figure: A tropical line



$F(X,Y) = \min(8,\ 4+X,\ 2+Y,\ 1+2X,\ X+Y,\ 1+2Y,\ 2+3X,\ 2X+Y,$
$X+2Y,\ 4+3Y,\ 8+4X,\ 4+3X+Y,\ 2X+2Y,\ 2+X+3Y,\ 4Y)$

Figure: A tropical planar quartic. (Joswig (2021), Figure 1.4)

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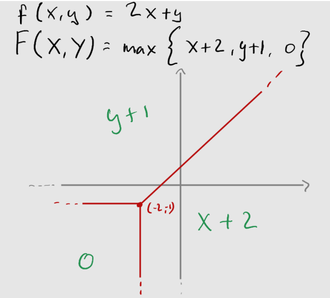

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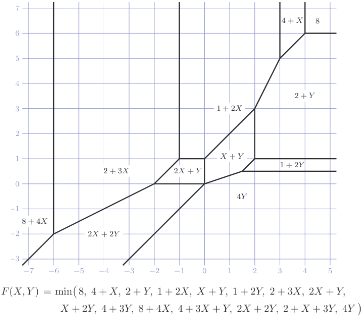

Figure: A tropical planar quartic. (Joswig (2021), Figure 1.4)

$\mathcal{T}(F_{ij})$ induces a polyhedral subdivision of $\mathbb{R}^E$. For generic C it holds that

$$\pi^{ij}_{\text{crit}}(\mathcal{G}, C) = \pi \iff C \text{ lies in the interior of the region of } \mathcal{T}(F_{ij}) \text{ corresponding to } \pi.$$

Describing $\mathcal{T}(F)$

If $F = Trop(f)$ constant coefficients, (Joswig, 2021, Corollary 1.21) implies that

$$\mathcal{T}(F) \text{ is the } \textit{Normal Fan} \text{ of } \mathrm{Newt}(f) \ ,$$

where for $f = \sum c_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots x_n]$, $\mathrm{Newt}(f) := \mathrm{conv}(\{\alpha \in \mathbb{Z}^n_{\geq 0}, c_\alpha \neq 0\})$

Example (Joswig (2021), Example 1.22)

Let $F(X, Y) = \max\{X, 2X, 2Y, X + 2Y, 2X + Y\}$.

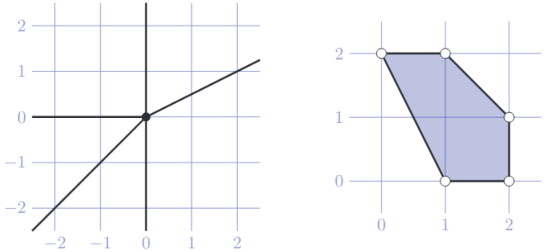

Figure: The rays of $\mathcal{T}(F)$, considered as a fan, are the inward pointing facet normals of the Newton polygon. (Source: Joswig (2021), Figure 1.7)

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Theorem (FN+25, Theorem 3.1)

The critical path fan $\mathcal{F}_{\mathcal{G}}$ coincides with the normal fan of the polytope $P_{\mathcal{G}}$, where

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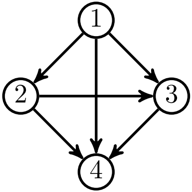
Furthermore: the unique critical path structures of $\mathcal{G}$ correspond bijectively to the *vertices* of $\mathcal{P}_{\mathcal{G}}$.

Example(sketch)

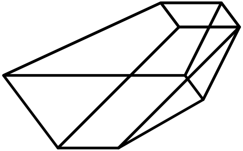
$\mathcal{G}$



$$C = \begin{pmatrix} -\infty & c_{12} & c_{13} & c_{14} \\ -\infty & -\infty & c_{23} & c_{24} \\ -\infty & -\infty & -\infty & c_{34} \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} \in \mathbb{R}^E \cong \mathbb{R}^6$$

$$\begin{aligned} F_{13}(C) &= \max\{c_{12} + c_{23}, c_{23}\}, \\ F_{24}(C) &= \max\{c_{23} + c_{34}, c_{24}\}, \\ F_{14}(C) &= \max\{c_{14}, c_{12} + c_{24}, c_{13} + c_{34}, c_{12} + c_{23} + c_{34}\}. \end{aligned}$$

$\mathcal{F}_{\mathcal{G}} \subset \mathbb{R}^6 =$ common refinement of $\mathcal{T}(F_{13})$, $\mathcal{T}(F_{24})$, $\mathcal{T}(F_{14}) = \mathcal{N}(\mathcal{P}_{\mathcal{G}})$.



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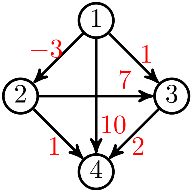

$$C = \begin{pmatrix} -\infty & -3 & 1 & 10 \\ -\infty & -\infty & 7 & 1 \\ -\infty & -\infty & -\infty & 2 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} \in \mathbb{R}^E \cong \mathbb{R}^6$$

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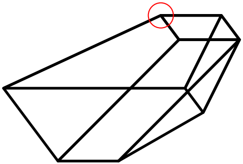
C lies in a full-dimensional cone of $\mathcal{F}_{\mathcal{G}}$ corresponding to a *vertex* of $P_{\mathcal{G}}$.



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Max-Linear Conditional Independence

Max-Linear Bayesian Networks (MLBNs)

Let $\mathcal{G}$ be DAG on n vertices with edge weights $c_{ij} \geq 0$ for $i \to j \in E(\mathcal{G})$.
A random vector $X = (X_1, \dots X_n)$ is distributed according to a max-linear model on $\mathcal{G}$ if

$$X_j = \bigvee_{i \in \mathrm{pa}(j)} c_{ij} X_i \vee Z_j, \qquad c_{ij}, Z_j \geq 0 \tag{3}$$

where $\vee = \max$, $\mathrm{pa}(j)$ is the set of parents of j in $\mathcal{G}$, and the Z_j are independent, atom-free, continuous random variables.

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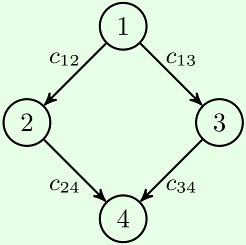
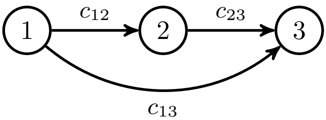


$$X_1 = Z_1$$
$$X_2 = c_{12} X_1 \vee Z_2$$
$$X_3 = c_{13} X_1 \vee Z_3$$
$$X_4 = c_{24} X_2 \vee c_{34} X_3 \vee Z_4$$

Challenges of the Max-Linear setting

The conditional independence structure of a MLBN depends on the choice of edge weights:



The CI statements of this model are $\begin{cases} \{1 \perp\!\!\!\perp 3|2\} & \text{if } \; c_{13} \leq c_{12}c_{23} \\ \emptyset & \text{if } \; c_{13} > c_{12}c_{23}. \end{cases}$

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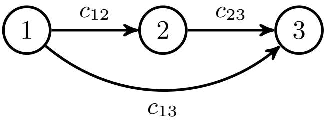

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This motivated the *C^*-separation* criterion of Améndola et al. (2022).

Maxoids

We refer to the set of conditional independence statements which hold in a MLBN on $(\mathcal{G}, C)$ as a *Maxoid* and denote it by $\mathcal{M}_*(\mathcal{G}, C)$.

Maxoids

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Let $(\mathcal{G}, C)$ and $(\mathcal{G}', C')$ be two weighted DAGs on the same node set and generic weights C and C'. Then

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⚠ We implicitly identify $(\mathbb{R}^E_{>0}, \vee, \cdot) \rightsquigarrow (\mathbb{R}^E, \vee, +)$ by taking logarithms.

The diamond revisited

The three maxoids associated to the diamond graph are

$\mathcal{M}_1 = \{[2 \perp\!\!\!\perp 3 \mid 1]\,, [1 \perp\!\!\!\perp 4 \mid 2,3]\,,\ [1 \perp\!\!\!\perp 4 \mid 2]\}$ $\quad c_{12} + c_{24} > c_{13} + c_{34}$

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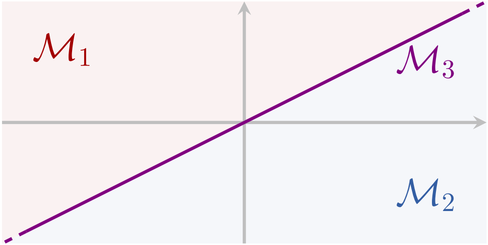

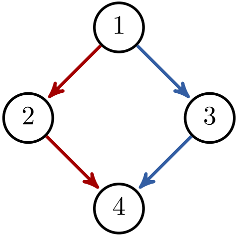

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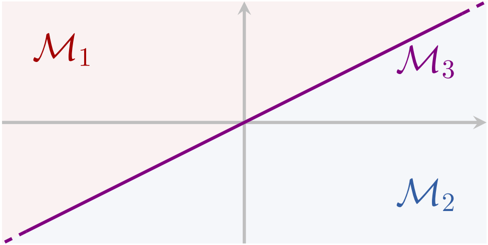

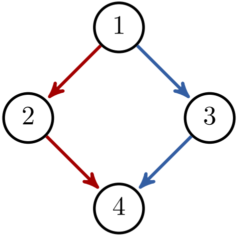

$\mathcal{M}_3 = \mathcal{M}_1 \cup \mathcal{M}_2$. This is not a coincidence!

The Maxoid Fan

Theorem (FN+25, Corollary 2.13)

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$$F_1 \textit{ is a face of } F_2 \implies \Phi(F_1) \supseteq \Phi(F_2) \quad \textit{for all } F_1, F_2 \in \mathcal{F}_\mathcal{G}.$$

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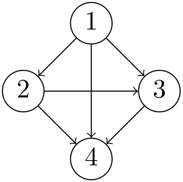
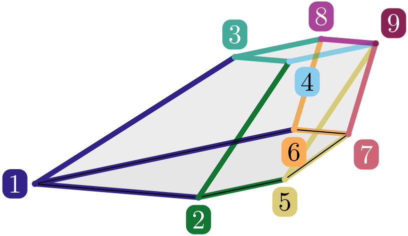
Bonus fact: the generic maxoids are in bijection with the vertices of $P_{\mathcal{G}}$. From now on, we refer to $P_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{G}}$ as the *maxoid polytope* and *maxoid fan* respectively.

Example: Continued





- Each colored vertex of $P_{\mathcal{G}}$ corresponds to a unique generic Max-Linear CI structure (Maxoid) associated to $\mathcal{G}$.

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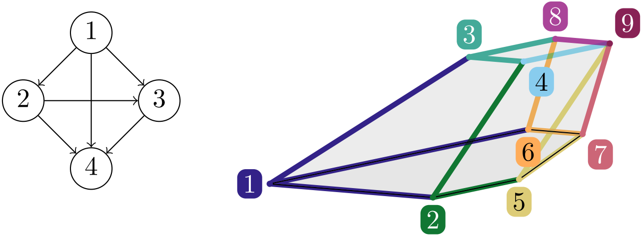
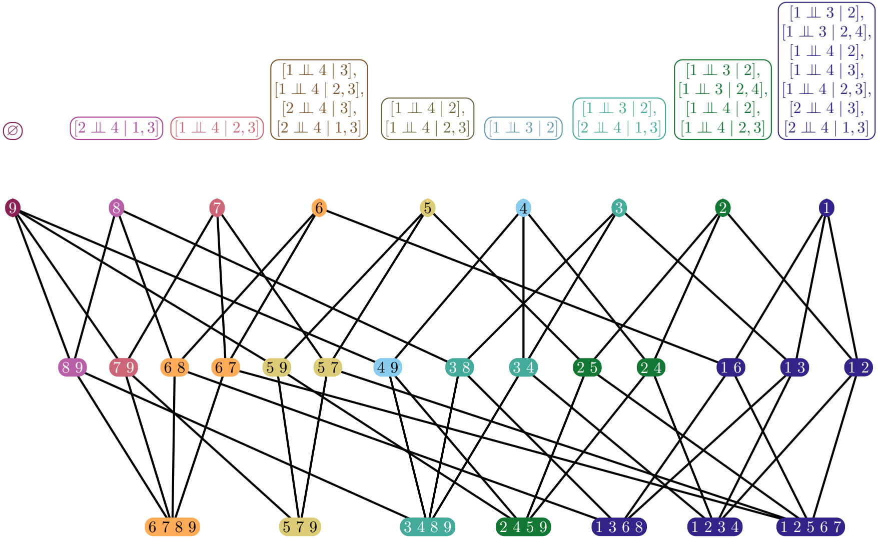


- Each colored vertex of $P_{\mathcal{G}}$ corresponds to a unique generic Max-Linear CI structure (Maxoid) associated to $\mathcal{G}$.
- Weight matrices corresponding to higher-dimensional faces of $P_{\mathcal{G}}$ give rise to maxoids which are unions of generic maxoids.

Example:  Continued



Applications

Application I: Enumeration of CI structures

The enumeration of all possible CI structures on n nodes requires computing them explicitly:

n	#TDAGs	#distinct maxoids	#distinct generic maxoids
3	3	4	4
4	18	41	40
5	181	987	892
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Conversely, the number of maxoids which can arise from a specific DAG $\mathcal{G}$ can be computed by constructing $P_{\mathcal{G}}$ and counting its vertices:

n	$\#E$	$\dim(P_{\mathcal{G}})$	#vertices $P_{\mathcal{G}}$
3	3	1	2
4	6	3	9
5	10	6	103
6	15	10	3324

Table: Data for the perfect DAG on n nodes (i.e. $E = \{(i,j) : 1 \leq i < j \leq n\}$), for $3 \leq n \leq 6$

Application II: CI implication

The *local CI implication problem* states:

If a set of CI statements S *holds in* $\mathcal{G}$ *, does* $[I \perp\!\!\!\perp J \mid K] \notin S$ *also hold in* $\mathcal{G}$*?*

This problem can be solved using polyhedral methods.

```
julia> using Maxoids, Oscar;
julia> G = complete_DAG(4);
julia> maxoid_implication(G, [ CI"24|13" ] => [ CI"14|3" ])
(false, [-Inf 1/4 0 1/2; -Inf -Inf 0 0; -Inf -Inf -Inf 0; -Inf -Inf -Inf -Inf])
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Figure: Code snippet from Boege et al. (2025), which queries a CI implication using our package Maxoids.jl (`https://doi.org/10.5281/zenodo.17737553`).

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Generally, Maxoids are *compositional graphoids* which additionally fulfill:

Amalgamation: $[i \perp\!\!\!\perp j \mid KM] \wedge [i \perp\!\!\!\perp j \mid LM] \implies [i \perp\!\!\!\perp j \mid KLM]$

Strong Spohn: $[i \perp\!\!\!\perp j \mid klM] \wedge [k \perp\!\!\!\perp l \mid iM] \wedge [k \perp\!\!\!\perp l \mid jM] \implies [k \perp\!\!\!\perp l \mid M]$

$[i \perp\!\!\!\perp j \mid klM] \wedge [k \perp\!\!\!\perp l \mid iM] \wedge [k \perp\!\!\!\perp l \mid M] \implies [k \perp\!\!\!\perp l \mid jM]$

Thank you! Questions?

https://arxiv.org/abs/2504.21068

Améndola, C., Klüppelberg, C., Lauritzen, S., and Tran, N. M. (2022). Conditional independence in max-linear Bayesian networks. *Ann. Appl. Probab.*, 32(1):1–45.

Boege, T., Ferry, K., Hollering, B., and Nowell, F. (2025). Polyhedral Aspects of Maxoids. arXiv:2504.21068.

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